TO PLEASE BOTH THE EAR AND THE EYE:
MOSES MENDELSSOHN, EQUAL TEMPERAMENT
AND THE DELIAN PROBLEM

An introduction to and annotated translation of
"Versuch, eine vollkommen gleichschwebende
Temperatur durch die Construction zu finden"

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In 1761 Friedrich Wilhelm Marpurg included in Volume V of his *Historisch-Kritischen Beyträgen zur Aufnahme der Musik* (Part 2, pp. 95–109) an essay giving a geometrical construction for equal-tempered division of the octave. In Marpurg's introduction to the essay he states that he is not at liberty to disclose the author's name, but in 1777, the index appended to Volume VI named Moses Mendelssohn as the author. In the intervening sixteen years, many had thought it to be the work of the theoretician Johann Philipp Kirnberger, and it is possible that Marpurg wanted to set the record straight by finally giving the credit to his friend Mendelssohn. Today there is no doubt that the attribution to Mendelssohn is the correct one.[1]

For Moses Mendelssohn — thinker, philosopher and epitome of the Jewish Enlightenment (Haskalah) of the time — this essay is unique. His interest in music does not reveal itself in his other writings, except for a generalized praise of the art in his aesthetic treatises. Mendelssohn was, of course, no stranger to music: his letters make a number of references to discerning attendance at concerts in Berlin, and he even took lessons

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1 The authorship question is fully discussed in an introduction by H. Borodianski in Mendelssohn (1931,1972: xxxvi–xli).

with Kirnberger in theory and keyboard playing. His keyboard skills never developed beyond the level of "playing a minuet slowly," according to the testimony of Nicolai, but he mastered music theory: he modestly reported of himself that he "knows ... all the musical proportions, chord inversions, various combinations of notes etc." Gerber, in his *Tonkünstler-Lexikon*, considered Mendelssohn "an outstanding music theoretician."[2]

An interest in music was never an unusual thing for scientists and philosophers, since the time of Pythagoras, if not earlier. Traditionally, music was included in the quadrivium, along with arithmetic, geometry and astronomy. The scientific revolution around 1600 attracted Kepler, Stevin (who proposed that equal temperament is not only useful but even "natural"), Beeckman and Descartes in musical investigations; later D'Alembert and Diderot the encyclopedists and Newton, Euler and other mathematicians addressed themselves to musical theory and accompanying problems.

Mendelssohn numbered among his friends most of the outstanding Berlin musicians of his time, including Agricola, Quantz, Kirnberger, Marpurg and others. It was from Kirnberger that he came to know the mathematical difficulties involved in equal temperament, and that these problems were considered vital to the development of music. They had been treated earlier — especially by Neidhardt and Werckmeister — but no completely satisfactory solutions to the problem of constructing an equal-tempered scale were yet available; practicing musicians and instrument builders used approximations or cut-and-try methods for dividing the octave which, good as they were, did not stand up to the rigorous scrutiny of mathematicians as being theoretically sound.

Mendelssohn knew mathematics; he had studied the subject along with philosophy, and was considered to be, along with Euler and Jacobi, one of Berlin's finest mathematicians. His lecture on probability, given before the leading intellectuals of his day, was firmly grounded in the mathematical theory of the subject.[3] The possibility of applying his mathematical knowledge to a musical problem must have appealed to his intellectual curiosity, for he agreed readily to Kirnberger's proposal to find a solution.

2 Mendelssohn 1931,1972: xxxix.
3 Mendelssohn 1931,1972:. xxxviii. Mendelssohn's notebooks also contain some mathematical writings, which demonstrate his practical approach to geometry. Altmann (1973) has a number of references to Mendelssohn's interest in mathematics.

The problem is simply stated: equal temperament requires that all the intervals between adjacent notes in a chromatic octave of twelve notes be equal, or in other words that all semitones, both chromatic and diatonic, be equal. In physical terms, this requires that there be a constant ratio between the lengths of strings sounding adjacent semitones (assuming the strings to be of equal thickness and density and under equal tension; the monochord is the usual metaphor for this). Since the ratio of the interval of an octave is 2:1, the ratio of the equal-tempered semitone must be the twelfth root of 2.[4] But it can be shown that this ratio cannot be constructed with unmarked straightedge and compass, the tools classically admitted in geometric constructions, so the problem is how to carry out the "impossible" construction.

This problem was known, in another guise, to the Greeks; it was treated by (among others) Newton, whose solution, like other solutions that had been proposed earlier, depended on violating the rules of straightedge-and-compass construction[5] by the artifice of making marks

4 This was first noted in Europe in the 1580s by Simon Stevin, the inventor of decimal fractions, in his "De spiegheling der singconst." (For a modern translation see "On the theory of the art of singing," in: *The Principal Works of Simon Stevin*, transl. A.D. Fokker, Amsterdam, 1966)

5 See "Ruler and Compasses," in Hudson (1953). The standard restrictions permit only (a) describing a circle (or an arc thereof) with a given point as center and with a given length as radius; and (b) drawing a straight line segment of any length through two given points. New points are won only as the intersections of two lines, or of a line and an arc, or of two arcs. The limitation to the use of straight lines and circles implies that the only instruments available are straightedge and compass; the straightedge may be of unlimited (but of course finite) length, and the compass may be opened as wide as desired.
These restrictions define canonical straightedge-and-compass construction, where the straightedge is not to be marked (a marked straightedge is called a ruler) and the compass is of the "collapsing" kind, not retaining its setting once the point has been lifted from the paper. The restrictions on the tools would seem to imply that a length cannot be transferred from one line to another, but in fact there is a simple construction under the strict rules which does just that; for this reason, constructions are commonly given implicitly assuming a non-collapsing compass which can act as dividers. Other constructions assumed because they are simple and well known include bisecting a line segment, erecting a perpendicular to a given line through a given point, drawing a parallel to a given line through a given point, and bisecting an angle.
In algebraic terms, it follows that a construction is possible if and only if the numbers which define the *quaesta* can be derived from the given elements by a finite number of rational operations (the four elementary operations of arithmetic—addition, subtraction, multiplication and division) and/or the extraction of square roots. Incidentally, it can be shown that compass alone or, if a single pre-existing circle be given, straightedge alone, or straightedge and "rusty" compass (whose setting cannot be changed—this is usually called an Einheitsdreher) can construct anything that is possible under the standard rules. On this, see Chapter 17 of Gardner (1979).

on the straightedge. It was Newton's method that Mendelssohn applied to the musical problem.

Mendelssohn's description of Newton's method (which was actually based on earlier methods, as Newton himself indicates) is in three sections. First, he presents the problem, and argues that approximations, while "seem[ing] equal-tempered even to the most sensitive ear," are not wholly satisfying to the geometer, who seeks proofs and rigor. On the other hand, a "true" geometric construction is impossible; Mendelssohn then suggests that the compromise of a "mechanical" construction (the term is Newton's) can please both the ear and the intellect.

Next he gives Hero's method for finding two mean proportionals between two given lengths, which he calls the simplest (Mendelssohn refers the reader elsewhere for its proof). For its practical convenience, however, Mendelssohn goes to Newton's method, reproducing it almost literally. He also gives a Euclidian proof of the validity of this construction.

Finally, step-by-step instructions for dividing the octave into twelve equal semitones are given, based on Newton's method augmented by the usual procedure for finding a length x so that $x:a = b:c$, where a, b and c are given, so that the equal-tempered division need be carried out only once and then constitutes a template for the division of a string of any length. These step-by-step instructions are presented in simple and practical terms for the draftsman, who is not necessaily knowledgeable of geometry or of the theory of geometrical constructions.

It is this emphasis on practicality which most characterizes Mendelssohn's essay. He is well aware that he has discovered nothing new but is "standing on the shoulders of giants," as did Newton; his concern is with "selling" the idea that the "mechanical" construction is both acceptable and feasible. He addresses many audiences: musicians, music theorists, instrument builders, mathematicians and draftsmen. His arguments are stated explicitly and convincingly but never become polemic (Mendelssohn reserved his public polemic for philosophical and social debates).

We cannot know the extent to which the construction described in this essay was actually put to use; one guesses that it had no practical application. Most instrument builders probably preferred — as they do today — to measure their string and pipe lengths and their fret placements according to previously calculated lengths (tables of these lengths were available), rather than to construct them. Mathematicians

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would be familiar with Newton without Mendelssohn's recommenda-
tions. As one of the arguments against the use of equal temperament is
that it is not "natural" (Stevin was alone in claiming that it is), musicians
and music theorists might well have regarded the essay as an implicit
defense of equal temperament; this was a time when equal temperament
was beginning to win the day on its way to eventual dominance in the
musical world anyway (not yet achieved; see fn. 30), but perhaps the
mathematical side of Mendelssohn's mind found here an opportunity to
join with the musical side.

Moses Mendelssohn
Essay on Finding an Exact Equal Temperament by Geometrical Construction[6]

For many years attempts have been made to express equal temperament
in [rational] numbers.[7] The impossibility of accomplishing

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6 The original German text is reprinted in Mendelssohn (1931/1972); the translation here is
from this edition. I would like to thank Prof. Gisele Luther of Case Western Reserve
University for her help in understanding the text; errors and infelicities in the translation
are mine alone.

7 Most early luthiers used rational approximations for placing the frets; the most popular one
(and quite a good one, well within the bounds of the accuracy needed if we consider the
stretching of a string when it is depressed by fingering) was 17:18 for the equal-tempered
semitone, as attested by Mersenne (*Harmonie Universelle*, III, 48), among others (see
Appendix I for another luthier's rule). This differs from the "true" equal-tempered value by
only 0.06%; the octave obtained from applying this ratio twelve times is just 0.73% too
small. Vincenzo Galilei endorsed this approximation as well (*Dialogo della musica antica et
della moderna*, p. 49). Mersenne (op. cit., p. 68) also gives another approximation: it
amounts to setting the major third — or, rather, 4 "equal" semitones — equal to
$\sqrt[8]{3}-\sqrt{2}$. This is geometrically constructible, since the only root involved is the square root
(see fn. 5); and as only the integers 2 and 3 are used the construction is a convenient one,
and these numbers echo Pythagorean theory. (One is reminded of the approximation of as
$3+\sqrt{2}$ / 10.) The octave obtained from this approximation is only 0.15% too small. For
other approximations see Appendix II.
The celebrated philosopher Thomas Hobbes published and "proved" a geometric
construction using Mersenne's second approximation ("the duplication of the cube,
hitherto sought in vain"). See Molesworth (1966: vol. 7, 59–68 and Plate 46 in vol. 11). The
quotation above is from the dedication (p. 3). As Hinnant comments, "Hobbes clearly
failed to perceive his own limitations as a mathematician" (Hinnant 1977: 25).
A curious approximate construction was published in 1743 by Daniel Strähle; for details see
Murray Barbour (1951: 65–68). See also Dudley (1987: 95–96).

this has, to be sure, been crystal-clear;[8] but we have made do with approximations to the true values while keeping the errors unnoticeable. A trained ear can detect uncommonly small differences, but even the best trained sense of hearing is not so sharp that it may not be deceived. It was easy to find a harmonic[9] ratio which will seem equal-tempered even to the most sensitive ear. A mathematical construction has seldom been considered. It is known that this is possible, and that the required lengths can thus be found without the slightest error; but this has been considered inconvenient, presumably because a purely geometrical construction[10] of the required lengths demands the use of higher curves,[10] which admittedly cause indescribable practical difficulties. Neidhardt,[11]

8 The impossibility of expressing roots in general (square roots excepted) as rational numbers was known to the Greeks: the classic "Delian Problem" of duplicating the cube — constructing a cube with double the volume of a given cube — is the best-known example, its solution involving as it does the cube root of 2 (see fn. 19). The impossibility of two other constructions were known to the Greeks: the trisection of an angle and the quadrature of a circle (construction of a square whose area is equal to that of a given circle). Proof of their impossibility was beyond Greek mathematics, however, and had to wait until the nineteenth century (see fn. 21). (The trisection problem is, mathematically, of the same nature as the Delian problem, while the quadrature problem's impossibility is due to the transcendency of)

9 I.e., rational. Mendelssohn here assumes a classical interpretation of the word "harmonic," one which goes back to ancient Greek theory and exemplifies the mutual relations of mathematics and music.

10 An unfortunate but common misnomer; the circle is just as much a "higher curve" as are the other conic sections — parabola, hyperbola and ellipse. Newton was aware of this, of course, and wrote:
But it is not the Equation, but the Description that makes the Curve to be a Geometrical one. The Circle is a Geometrical Line, not because it may be expressed by an Equation, but because its Description is a Postulate. It is not the Simplicity of the Equation, but the Easiness of the Description, which is to determine the Choice of our Lines for the Construction of Problems. For the Equation that expresses a Parabola, is more simple than That that expresses a Circle, and yet the Circle, by reason of its more simple Construction, is admitted before it. The Circle and the Conick Sections, if you regard the Dimension of the Equations, are of the same Order, and yet the Circle is not numbered with them in the Construction of Problems, but by reason of its simple Description, is depressed to a lower Order, viz. that of a right Line ... Equations are Expressions of Arithmetical Computation, and properly have no Place in Geometry ... ["Appendix for the Linear Construction of Equations," in D.T. Whiteside, *Universal Arithmetick: Or, A Treatise of Arithmetical Composition and Resolution*, 226–227 (*The Mathematical Works of Isaac Newton*, vol. 2), New York/London, 1967 (facsimile of the London 1728 edition)].
A better, albeit somewhat awkward, term for "higher" in this context might be "less simple of construction."

11 Johann Georg Neidhardt composed a little treatise called *Beste und leichteste Temperatur des Monochordi, vermittelst welcher das heutiges Tages bräuchliche Genus Diatonico-Chromaticum also eingerichtet wird* (Jena, 1706); but here Mendelssohn quotes, somewhat imprecisely, Neidhardt's *Sectio canonis harmonici* (Königsberg, 1724).

who has done much for temperament, states the following: "As to geometrical construction: One geometric mean line is found by means of a line and a circle, but two [means] require circle and parabola, circle and asymptotic hyperbola, circle and hyperbola, or ellipses, infinities, etc. All this does not concern us," Neidhardt adds, "because it is much, much more convenient to make use of arithmetical approximations by the canone harmonico, inasmuch as the ear is thus satisfied, although Reason not at all."

But how? If both the ear and the mind can be satisfied, and easily at that, is it not even easier when the ear alone is satisfied through arithmetical approximation? A geometric construction of the mean of two given lines cannot be accomplished without the help of higher curves, which have their difficulties; but there exists a kind of construction called mechanical, which is easily carried out and which is just as exact as the geometrical one.[12] The geometer rejects it not for its inaccuracy but out of geometric obstinacy. He does not want to seek anything blindly, nor to use an instrument blindly and afterwards to see if he has used it correctly; he would rather always know in advance where to find what he desires and exactly to which point to bring the instrument. But with the so-called mechanical construction one often has to place the instrument at random and then move it to and fro until the proper place for it is found. If one does not wish to be obstinate, one can make use of a mechanical construction — and I believe that musicians have little reason to be obstinate. At least one can try it and see whether the required equal temperament can be found far more easily and more exactly than by use of the common arithmetical approximation.[13] I shall first set out separately the mathematical bases which prove the correctness of the construction, and afterwards specify, expressly and briefly, the rules the mechanical draftsman is to follow.

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12 Newton (1728): "that [construction] is *Geometrically* [italics original] more simple which is determined by the more simple drawing of Lines...I am here sollicitous not for a Geometrical Construction, but any one whatever, by which I may the nearest Way find the Roots of Equations of Numbers." Nowadays the appellation "geometrical" is often applied to constructions which Newton (and Mendelssohn) termed "mechanical"; those which make use of the straightedge and and compass according to the strict rules (see fn. 5), in Newton's terms "geometrical," are today often called "Euclidian."

13 The reference is probably to the 17:18 ratio mentioned above (fn. 7).

For equal temperament consists of thirteen strings of equal thickness and under equal tension,[14] of which the last is the octave of the first, and the others, which stand at equal distances one from the other, produce the same interval; i.e., they stand in the same ratio one to another. This is achieved when 13 lengths are found which follow a constant ratio,[15] with the ratio of the first to the thirteenth being as 2 to 1. For when this is the case, all the strings, equidistant[16] one from another, will give the proper intervals, with the last [string] being the octave of the first.

Let the first be C and the thirteenth c;[17] to be found then are eleven mean-proportional lines a, b, d, e, etc. between C and c, such that C:a = a:b = b:d = d:e = e:f = f:g = g:h = ... = l:m = m:c. Then the first is to the seventh as is the seventh to the thirteenth; furthermore, the first is to the fourth as is the fourth to the seventh, and the seventh is to the tenth as is the tenth to the thirteenth. So the seventh is found as he mean of the first and thirteenth, the fourth as the mean of the first and seventh, and the tenth as the mean of the seventh and thirteenth. Musicians call the first C, the fourth d$^{\#}$, the seventh f$^{\#}$, the tenth a and the thirteenth c.

The construction of these lines involves the use of straight lines and circles, and [so] is purely geometrical:[18] For let AB be the length of the C

14 Mendelssohn here assumes that the 13 strings are all of equal thickness and density, which will be the case if they are all cut from one uniform wire, or, what amounts to the same thing, if "strings" is taken to mean lengths on a monochord.

15 This ratio is, of course, the twelfth root of 2 (1.05946 +), but Mendelssohn will speak of it in terms of mean proportionals, which in the case of only one mean proportional between two lengths is simply constructed.

16 Musically, that is.

17 A certain confusion for the reader begins here: Mendelssohn uses letters of the Latin alphabet both for note-names and for points in his diagrams and explanations of them. These might have been better differentiated and better understood if he had reserved the capital letters for the geometric points and the lowercase letters for note-names, with, say, c' for the higher c; another solution would be to use Greek letters for the geometric points. Furthermore, Mendelssohn follows classical models in denoting both a note and the string producing that note by the same letter; this practice is not so confusing, but it is not always unambiguous. The scale in Mendelssohn's notation is, then: C cis d dis e f fis g gis a b h c (note that Mendelssohn uses uppercase C for the lowest note of the octave which continues d, e, ...and ends with lowercase c). I have transcribed this as: C c$^{\#}$ d d$^{\#}$ e f f$^{\#}$ g g$^{\#}$ a b^{b} b c.

18 By "these lines" are meant the lines for d$^{\#}$, f$^{\#}$ and a, dividing the octave into four equal parts. The construction for a (single) mean proportional between any two given lengths is well known. Mendelssohn here gives the usual method, giving f$^{\#}$ between C and c, and adds to it a second stage which produces d$^{\#}$ as the mean proportional between C and f$^{\#}$ on the one hand, and a as the mean proportional between f$^{\#}$ and c on the other.

string (Fig. 1[19]). Describe the semicircle ADEB with the center c, and erect at c the perpendicular cD. Draw the line AD; then AB:AD = AD:Ac. Mark off AF equal to AD and erect the perpendicular FE. Draw AE; then AB:AE = AE:AF. Draw on AF the semicircle AHF with center G; erect the perpendicular cH and draw AH: then FA:AH = AH:Ac. All this is in accordance with known and proven geometrical methods. Thus AB is the length of the C string, AE of the $d^\#$ string, AD or AF the length of the $f^\#$ string, AH the length of the a string, and Ac or cB or cD the length of the c string; and these stand in constant ratio, so that $C\text{–}d^\# = d^\#\text{–}f^\# = f^\#\text{–}a = A\text{–}c$, since:

AB:AE = AE:AF
therefore $AB{:}AF = AE^2{:}AF^2$
furthermore AF:AH = AH:Ac
therefore $AF{:}Ac = AH^2{:}Ac^2$
Also, it will be true that:
AB:AF = AF:Ac (since AF = AD)
then $AE^2{:}AF^2 = AH^2{:}Ac^2$
consequently AE:AF = AH:Ac

Therefore:
$AC(=C){:}AE(=d^\#) = AE{:}AF(=f^\#) = AE{:}AH(=a) = AH{:}AC(=c)$,
which was to be proved.

We have accomplished the first and easiest step, namely finding three means between C and c. Now the difficulty is in the task of dividing each of these intervals with two mean proportional lines, thus making from the five constant-proportional lines [already] found, thirteen. For if all members of a constant progression are divided into the same number of mean proportional of terms, then these terms also form a [single] geometric progression, as the following shows. Given that a:b = b:c, and that the number of constant-proportional terms inserted between a and b, and also between b and c, is = m; let the last term before b be e and the first term after b be f, then we have:

$a{:}b = e^{m+1}{:}b^{m+1}$
furthermore $b{:}c = b^{m+1}{:}f^{m+1}$
now since a:b = b:c, then $e^{m+1}{:}b^{m+1} = b^{m+1}{:}f^{m+1},$
and so e:b = b:f and the progression proceeds unbroken.

So if we could divide each of the four intervals already found with two mean proportionals, we would have the required thirteen lines, and therefore also the equal temperament.

19   For Mendelssohn's figures, see Appendix III.

It depends, then, solely on the well-known problema deliacum,[20] which made such a stir in ancient times. Plato, Hero of Alexandria, Philo [of Byzantium], Apollonius [of Perga], Diocles, Pappus [of Alexandria], Sporus [of Nicaea] and Erathostenes all proposed solutions at various times, which can be found in Eutocius and in Sturm's German translation of Archimedes' works.[21] These great men found only

20 The Delian Problem gets its name from a legend related by Philoponus: The Athenians were afflicted by a plague of typhoid fever, and consulted the oracle at Delos for a solution to their woes. Apollo replied that his altar, which was cubical, must be doubled in size, whereupon they doubled its edge; the plague intensified instead of abating. It then became clear that the meaning of the oracle was to double the cube's volume, and thus the hapless citizens were faced with the geometrical problem, which was shown by Hippocrates (in the late fifth century) to depend on finding two mean proportionals. It is sometimes called the Delphic Problem.

21 Mendelssohn's mention of Eutocius refers to the latter's commentary on Book II, Prop. I of Archimedes' *Commentaria in libros de sphaera et cylindro, dimensionem circuli, de planorum aequilibris* (referred to below as *On the Sphere and Cylinder*), which is called by Knorr (1985) an "anthology of cube duplications." For a modern edition of Eutocius' commentary, see *Commentaria in libros de sphaera et cylindro, dimensionem circuli, de planorum aequilibris*, transl. J.L. Heiberg, in *Archimedis opera omnia cum commentariis Eutocii*, vol. 4, Leipzig, 1915. Eutocius of Ascalon is dated to the end of the fifth or beginning of the sixth century CE.
Plato: The ascription of a certain solution to Plato has its origin, apparently, in a passage in Eutocius; the solution he cites is probably a misattribution of Eudoxus, known as a disciple of Plato. However, Eratosthenes' Platonicus, as reported by Plutarch (*Moralia*, 718e), has Plato demanding a formal solution as opposed to the mechanical constructions which had been devised.
Hero (late first century CE): In Hero's *Belopoeica*, he attributes this solution to the Alexandrian Ctesibius, but it may also have been inspired by Philo.
Philo (third century BCE): Philo too wrote a work called Belopoeica in which he presented a solution. A slightly different and more formal solution is attributed to Philo by Eutocius; it is this solution which Zarlino repeats ("Consonantis Diapason in Duodecim Semitonia equalis divisio," *Suppl. Mus.*, p. 211).
Apollonius (around 200 BCE): The main source for Apollonius' method is Eutocius; it is also known in Arabic translations.
Diocles (around 100 BCE) gives a solution using two parabolas, in his Prop. 10. See Toomer (1976: 90–97). Diocles' solution using the cissoid (see fn. 23) is in his Prop. 15 (Toomer 1976: 235–243). Eutocius has Diocles' solutions in a somewhat altered form; he does not mention the author of the two-parabola solution, who is generally assumed to be Menaechmus; Mendelssohn undoubtedly refers to the solution which uses the cissoid.
Pappus (early fourth century CE): See Book VIII of the *Collectionis quae supersunt*, Pappus (1986–1987, 1965).
Sporus (late third century BCE): Only Eutocius' account is known.
Eratosthenes (around 250 BCE): This too is known through Eutocius' catalog of solutions,
here in the unusual form of a letter from Eratosthenes to the king Ptolemy.
Eutocius' accounts of solutions by Menaechmus, Archytas and Nicomedes are not mentioned by Mendelssohn.

mechanical methods; a geometric construction of two mean proportionals is impossible without the help of higher curves. [22] Nicomedes was the first to invent the conchoid for this purpose; [23] later other higher curves were used as well. Since we are not going to deal with higher curves, but rather must carry out the construction with circle and

The traditional classification of solutions divides them into "linear," "plane," "solid," and construction by "neusis" ("verging"; Lat. "inclinatione"). In these terms, there is no "plane" (straightedge and compass) solution to the Delian Problem. The constructions listed by Eutocius can be classified as follows:

"Solid": Menaechmus (intersections of parabolas and hyperbola); Archytas (intersections of cone, cylinder and torus);

"Linear": Diocles; Nicomedes (both using "higher" curves; see fn. 23);

"Neusis" (in these constructions this means swinging a marked straightedge): Hero; Philo; Apollonius; Pappus; Sporus;

Mechanical devices (actually a form of "neusis"): Plato; Eratosthenes.

Archimedes showed that the finding of two mean proportionals is entailed in the problem "Given a cone or cylinder, find a sphere equal [in volume] to the cone or cylinder" *(On the Sphere and Cylinder* II, Prop. 1).

A fairly comprehensive overview of these solutions is found in Knorr (1985). Heath (1921) gives a brief account of the solutions in antiquity. Earlier histories of the Delian Problem include *Historia problematis de cubi duplicandi* by N. T. Riemer (1798) and another book of the same title by C. H. Biering (1844).

The Sturm referred to by Mendelssohn is of course not Ambros Sturm, who published an excellent but neglected work titled *Das delische Problem* (Linz, 1895–97), but rather Johann Christoph Sturm, with his *Des unvergleichlichen Archimedes Kunst-Bücher oder heutigs Tags befindliche Schriften* (Nürnberg, 1670).

22 A square root is easily constructed, as it is the mean proportional between 1 and the number whose root is sought, but no other prime root admits a geometrical straightedge-and-compass construction. An elementary proof of this assertion is to be found in Chapter I and the first section of Chapter II of Klein (1956); for a more rigorous proof see Meschkowski (1966) or Jones et al. (1991).

The impossibilities of cube duplication and of trisection of the angle were proved by Wantzel in 1837, and that of circle quadrature by Lindemann in 1882; but the society of would-be angle-trisectors, circle-squarers and cube-duplicators is still alive, as is most amusingly documented by Dudley (op. cit.). And in Mendelssohn's time, in 1775, the Paris Academy found it necessary to pass a resolution that no more "solutions" to the three classic problems were to be examined, so as not to waste the time of the Academy (*Histoire de l'Academie royale*, année 1775, p. 61). See also fn. 7, on Thomas Hobbes.

23 The conchoid (or cochloid) of Nicomedes is a two-branched curve of the fourth degree with both branches having a given straight line as their asymptote. Originally defined mechanically, as a locus, it has as its analytic expression $(x^2 + y^2)(x - a)^2 - b^2x^2 = 0$, where $x = a$ defines the asymptote and b is a parameter which determines the precise shape of the curve. In polar coordinates the expression is simpler: $r = k + a \sec\theta$. Nicomedes also invented a mechanism or tool for tracing the conchoid. Klein (op. cit., p. 47) shows how the conchoid can be used to solve the problem of the trisection of an angle; for the Delian problem he uses the third-order cissoid of Diocles (Diocles himself did not use the name cissoid for this curve, nor did Eutocius; but this is the name common since the seventeenth century): $x^3 + (x - a)y^2 = 0$, or in polar coordinates $r = k + a \sin\theta \tan\theta$.

[straight] line alone; therefore the solution of Hero, among all those reported by Eutocius, seems the simplest. Yet I shall add to this another construction, to be found in Newton's *Arithmetica Universali*,[24] which seems to me the most convenient of execution.

Hero's instructions are carried out as follows: Suppose we want to find two mean proportionals between C and d#. Describe the lines AB (= C) and AE (= d#) perpendicular to one another (Fig. 2)[see App. IIIa], and complete the rectangle ABFE. Draw AF and BE, which bisect each other at D. Place a straightedge on E and rotate it to and fro around this point until DC becomes equal to DG. Now draw the line CEG; then CF and GA are the required mean proportionals, and therefore FC is the length of the c# string, and GA the length of the d string. The proof can be found in Sturm.

If one proceeds with d# and f#, with f# and a, and with a and c as previously was done with C and d#, then one obtains e and f, g and g#, b♭ and b — all the required lengths. This method is called mechanical because the straightedge cannot be placed with certainty; one must first seek a location for it with DC = DG. But it is easily seen that this fact does not prevent the method from being correct.

Newton, in his *Arithmetica Universali* (Appendix de Aequationum constructio lineari), gives a number of other equally mechanical constructions for this same problem, of which the following appears more convenient than Hero's method.

He bisects the line AB, the first of the two given lines (in our case = C), at E (Fig. 3). With center at A and radius AE he describes the circle EC, through which the second given line EC (in our case = d#) passes as a chord. He then produces the lines EC and BC, without specific limits. He puts the straightedge on A, and swings it to and fro between the lines so produced until the segment GF becomes equal to AE or to EB, and he draws the line FGA. When this is done, he states, then CF and AG are the two mean proportionals between AB and EC; in our case CF = c# and AG = d. Constructia nota est, Newton concludes.

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24   The reference is to Newton's *Arithmetica universalis; sive de compositione et resolutione arithmetica liber*, of which Mendelssohn possessed copies of both the first edition (Cambridge, 1707) and the third (Leyden, 1732). [References to this work are given here to the English edition; Whiteside (1967).] Mendelssohn does not, as we see, claim to have invented anything; his contribution is in adapting known methods to the musical problem of equal temperament and presenting the construction in an easily reproduced form. Vieta gave essentially the same method in his *Opera mathematica* (Leyden, 1646), pp. 393–396.

I will be permitted to demonstrate what Newton assumed as obvious.[25] Great geniuses attain in one step an end which ordinary mortals need a whole series of keys to decipher. The proposition was this: Describe a circle with A as center and with radius AE = EB, and extend the chord EC and the line BC indefinitely; let FA be placed so that FG = AE; then it follows that AB:CF = CF:GA = GA:CE.

Proof:

Complete the circle, and produce FA to H (Fig. 4); draw AK through A parallel to EC. Since AK∥EC, it follows that
BA:BE = AK:EC.

Now BE = ½AB; therefore also EC = ½AK.
Furthermore, the triangles FGC and KGA have equal angles (since FC is by construction ∥KA); so
CF:FG = KA:GA
and
CF:2FG = ½KA:GA

But 2FG = AB (per hypoth.), ½KA = CE (per demonst.); therefore CF:AB = CE:GA, and by inversion AB:CF = GA:CE. Similarly, AB+GA:CF+CE = AB:CF = GA:CE. Now AB+GA = FH; so AH+FG = AB and CF+CE = FE, therefore FH:FE = AB:CF = GA:CE. Furthermore (by Euclid[26], prop. XXXVII, L. 3),

FH:FE = FC:FL.

Then since FL = AG, and AL = FG (per hypoth.); so FH:FE = FC:AG. Therefore, it was previously shown that FH:FE = AB:CF = GA:CE, and so CF:AG = AB:CF = GA:CE. And finally, AB:CF = CF:AG = AG:CE. Which is the proposition which was to be proved.

Find in exactly the same way the two mean proportionals between d# and f#, between f# and a, and between a and c: one thus wins e and f, g and g#, and bb and b. But this can be more easily done. For once C and c# are found, the distance C-c# is known. And since C:c# = c#:d, so also C-c#:C = c#-d:c#. The same ratio pertains for d-d#:d, d#-e:d#, e-f:e, f-f#:f, f#-g:f#, g-g#:g, g#-a:g#, a-bb:a, bb-b:b, and b-c:b, which can all be shown in the same way. So one has but to erect a perpendicular with the

25   This slightly reproving compliment to Newton is misleading and somewhat unfair. Newton did of course give a proof of the construction, albeit indirectly, earlier (on pp. 230–231). There he presented a construction for solving the reduced cubic equation $x^3 + qx + r = 0$. The two-mean-proportionals problem is a special and a simpler case of this; its proof follows immediately from the proof of the more general case.

26   Mendelssohn used the Oxford (1703) edition of *Euclidis quae supersunt opera omnia*.

length of the distance C-c# on the C string (Fig. 6), and complete the triangle; the rest of the distances c#-d, d-d#, d#-e, e-f, etc. are then quite easily found, and the last distance b-c will, if all has been properly observed, fall on the point c, as the midpoint of the large string, as any beginner in mathematics knows.

I believe I have said all that is necessary for understanding the proposed construction. A draftsman can take this on faith, if he does not wish to be concerned with the mathematical basis. But he must exercise all possible care to execute all exactly as directed. I shall try to make his work as brief as possible, and also give directions which will show the way, if he is sufficiently careful.

Describe the semicircle ADEB on the segment AB with C at its center (Fig. 1) [see App. IIIb], erect the perpendicular CD and draw AD. Mark off AF with the length of AD, erect the perpendicular FE, and draw AE. On AF describe the semicircle AHF with the midpoint G as its center, erect the perpendicular CH, and draw AH (Fig. 5). Then mark off AE from A for d#, AD from A for f#, AH from A for a, and AC from A for c.

Now bisect AB at E (Fig. 3)[27] and describe the arc EC with radius AE and center A; mark off the length of d# (from Fig. 5) from E to C, producing this line as far as may be necessary. Similarly produce a line indefinitely from B through C. Then place a straightedge on A, and move it to and fro until the segment FG on the straightedge, contained between the two produced lines, is exactly as long as AE or AB; then draw the line FGA.

The two lines CF and AG are then marked off (in Fig. 5) from A for c# and d. If desired, both lines, CF and GA, may be found from Figure 2, according to the instructions previously given. One will thus be reassured on finding that the segments GA and CF are identical in both constructions; but it is not necessary to go to such an extent if one takes sufficient care in fitting FG.

Once the distance C-c# is found as accurately as possible, any draftsman will know how to find the others without tediously redrawing Fig. 3 four times with the same care. [To this end,] draw the line CD (Fig.

27   This is the same as Newton's Figure 99 (which itself is only a simplification of his Figure 92), with the following substitutions in the names of points

Newton:	K	A	C	X	E	Y
Mendelssohn:	A	B	E	C	G	F

6) with the length of C, erect the perpendicular CE = C-c#, and draw ED. Mark off CA with the same length C-c#, and erect the perpendicular AB, which will be = c#-d. Similarly, mark off AF with the length AB and erect the perpendicular FG, which will be = d-d#; in the same manner obtain f-f#, f#-g, g-g#, etc.

Since the points d, d#, f#, a and c have already been determined in the previous construction, one has now an infallible test of whether or not the instructions have been accurately followed. Since here only the distance C-c# has been used, along with the triangle CDE (Fig. 6), all the rest are found without difficulty; so one sees immediately whether or not the points d, d#, f#, a, and c of Figure 6 match the d, d#, f#, a and c found otherwise, in Figure 5. If so, the draftsman can be sure that he has determined a perfect equal temperament, and one which satisfies not only the ear, but also the intellect, insofar as our hands and tools can accomplish. Otherwise, on the contrary, he sees clearly that he has deviated from accuracy, and must start anew.[28]

Once a single line has been divided in this proportion, then any other length, be it shorter or longer than the one already divided, can be similarly divided without undue trouble. Given a line XW divided in equal temperament (Fig. 7), and required to divide a shorter line XY or a longer one XZ in the same proportions. Attach XY or XZ to XW at any arbitrary angle YXW or WXZ. Then draw parallels from C, c#, c, c#, etc., and any shorter line XY or longer one XZ will be identically divided into equal temperament, as is easily seen.[29]

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28   So there can (and probably will) be some inaccuracies. Mendelssohn makes no mention of the errors which can creep into the construction owing to the thickness of the lines drawn, but speaks only of the inaccuracies of the draftsman himself. Yet the errors which might be introduced by assuming a line to have zero thickness are probably as great as that of the 17:18 approximation (see fn. 7), if not greater: the error of this approximation will reach only 0.3 mm (a reasonable thickness for a carefully drafted line) in 50 cm. A comparison of Newton's construction with that of Hero will show that Newton's method has smaller angles, and so the points of intersection of the lines forming these angles are less clearly defined. Nevertheless, with the Newton/Mendelssohn construction "the mind has been satisfied," and this is what was important to Mendelssohn the philosopher and mathematician.

29   This paragraph and the accompanying Figure 7 are found only in Kirnberger's reprint (*Construction der gleich-schwebenden Temperatur*, ed. Johann Philipp Kirnberger, Berlin: Birnstiel [n.d., but before 1767]) of the essay; this edition also omits Marpurg's introduction. The added paragraph might possibly be attributed to Kirnberger himself.

INTRODUCTION

[by F. W. Marpurg]

The debate concerning the advantages of one unequal temperament or another has ended since Neidhardt began to acquaint us with equal temperament.[30] It has been found that none of them is useful, and that in keys where the small and large semitones[31] are to have the same size, all the notes between 1 and 2 must follow a geometric proportion. Much effort has been spent in calculating an equal temperament in various ways, e.g., (1) by means of extracting roots; (2) by equating the cycles of fifths and fourths; (3) by a geometric division of the ditonic comma; (4) by an arithmetic division of the same; (5) by a geometric division of the syntonic comma into eleven parts; (6) by an arithmetic division of the

30 Hopeful but of course untrue. The debate continued, with aestheticians and practicing musicians joining theoreticians in championing or denouncing equal temperament, whose proponents won, but only *de facto*, not before the last half of the nineteenth century. Opponents usually cite the harshness of the mistuned thirds and the irrationality of the ratios as against the chords and ratios of one variety or another of just intonation; they also bemoan the loss of differences between keys and modes in equal temperament. Here are a few examples. Francois Bédes de Celles (*L'Art du facteur d'orgues*, 1776-1778): "With the new [equal] temperament, all the keys being the same, they express everything equally, and there is nothing to compensate for the harshness of the thirds" (Steblin 1981: 65–66). Charles Earl Stanhope ("Principles of the Science of Tuning Instruments with Fixed Tones," *Philosophical Magazine* 25 [1806]: 17): "In the Equal Temperament ... everything is discord." John Broadhouse (*Musical Acoustics*, London: Reeves [around 1840]): "Equal temperament ... can never be used for any scale which can be called musical, according, at any rate, to the demands of modern harmony, as developed since the time of J. S. Bach ... If the semitones are [tempered equally] ... music would be impossible" (p. 363). Daniélou (1943: 219–220): ".. the artificial temperate scale [equal temperament] ... has twisted the development of modern musical thought in ... a strange direction." Much the same is the criticism of Levarie and Levi (1968). But equal temperament is now the rule; Arthur von öttingen (*Das duale Harmoniesystem*, Leipzig, 1913; cited in Rummenhöller [1967]), has this to say: "unser Ohr alles an einer reinen Stimmung Fehlende ergänzt ... keine noch so lange übung und Gewohnheit macht es uns möglich, ohne Klavierbegleitung temperiert zu intonieren. Daraus folgt, dass die reine Stimmung [ist] der Untergrund unserer höheren Empfindungen, psychischen Vorstellungen und musikalischen Wahrnehmungen."
    At the time of Mendelssohn's essay, Marpurg favored equal temperament, while Kirnberger was opposed (see fn. 32). Marpurg's answer to the objections was. "muss der Componist den Charakter seines Tonstücke, die Ausbildung einer Leidenschaft, die Kraft des Ausdrucks, aus ganz andern Quellen als aus der schöpferischen Kraft des Stimmhammers oder Stimmhorns herholen" (*Versuch über die musikalische Temperatur*, Breslau, 1776, p. 194). For an introductory discussion of some of the unequal temperaments proposed and/or used, see Bailhache (1989).
31 I.e., chromatic and diatonic.

same into the same number of parts; (7) by use of the rational numbers of the major chord 6:5:4:3 (Kritische Briefe über die Tonkunst, sections 39–41); (8) by an arithmetic partition of the diesis 125:128; (9) by a geometric division of the same; (10) by an arithmetic division of the smaller comma of the third 625:648; (11) by a geometric division of the same; and so forth.[32]

Although an equal temperament reckoned in such a manner gives the ear all possible satisfaction, yet because of the break[33] appearing at the end one cannot claim that any one of them completely satisfies the eye [as well], except for number 7. I ignore the differences found in the last [decimal] places owing to the varied methods of solution, if one rejects these. Mr. Kirnberger, himself one of our finest composers, was aware of

32   The ditonic comma (Marpurg's #3) is the ratio of six major tones (8:9) to the octave (1:2), or 524288:531441. The syntonic comma (#5) is 80:81, or the ratio of two tones of 8:9 to the major third 4:5. The diesis (#8) 125:128 (commonly called the lesser diesis) is the ratio of the octave to three major thirds (4:5). The smaller comma (or greater diesis) 625:648 (#10) is the ratio of four minor thirds (5:6) to the octave. Another small interval, not mentioned by Marpurg in this list, is the schisma (32768:32805), the ratio of the octave to four major tones plus a major third, or of the syntonic comma to the ditonic comma(see App. IIIb for an indirect mention of this).
      Dividing the syntonic comma into eleven parts (#5) is roughly equivalent to dividing the ditonic comma into twelve parts (#3), as the ratio of the two commas is about 11:12 (21.506 cents to 23.460 cents).
      The 6:5:4:3 ratio (#7) is used by Schröter; it is reported by Marpurg (*Versuch über die musikalische Temperatur*, pp. 179 ff.), along with many other temperaments.
      An arithmetic division of an interval leads to unequal steps; for example, 16:18 (= 8:9) is divided arithmetically into 16:17 and 17:18. Musically equal steps are obtained by geometric division; equal temperament is sometimes called geometric temperament.
33   Probably an allusion to the non-closure, after 12 steps, of the "cycle" of Pythagorean fifths and analogous "gaps" in these methods (the "error of closure" amounts to a ditonic comma; see n. 32).
34   Kirnberger, in vol. I of his *Die Kunst des reinen Satzes in der Musik* (Berlin, 1771), gives the good news first: "[With equal temperament] It is possible to play with almost complete purity in all the major and minor keys." But then he continues:
          First of all, it is impossible to tune in equal temperament without a monochord or something that takes its place. Consonant intervals can be tuned pure by the ear alone, but the dissonant ones cannot be found precisely. Second, the diversity of keys is eliminated by equal temperament... Thus nothing was...gained;...a great deal was lost [p. 11].
      Kirnberger's second objection to equal temperament refers to the loss of differences in the characters of the modes, arising from the fact that in non-equal temperament "the quality of the intervals changed whenever a different note was used as a final" [p. 3]. (He expands on these differences in vol. II (1779), pp. 70–76.) The third point he makes is that intervals should be such that a melody proceeds in "pure" intervals insofar as possible. [The translation and page numbers here are those of the English version (Beach and Thym 1982).]

this imperfection in our equal temperament [34] and wished to see an equal temperament for the monochord which would please ear and eye alike; he happened to read what Neidhardt wrote in his *Sectio canonis harmonici* concerning geometric constructio with regard to temperament. He took the opportunity to discuss this with an astute mathematician whose name I am not at liberty to give, and to ask him whether what Mr. Neidhardt only touched on superficially might not be examined more closely; and perhaps a practical rule could be formulated which would be more satisfactory than the arithmetical approach. Mr. Kirnberger's learned friend took the project upon himself, and after a little effort had the pleasure of solving the riddle and of closing the gap left by Mr. Neidhardt. Here is his essay on the subject, which so honors him by its excellent insights that it will please not only adepts of equal temperament but also mathematicians.

APPENDICES

Appendix I. A Second Luthier's Rule

Instead of the common 17:18 approximation to the equal-tempered semitone, some builders and players of fretted instruments used another rule-of-thumb method of division, not so accurate but perhaps easier to implement. The procedure has six steps:

1) Bisect the string; this gives the octave, with the entire length being the prime.
2) Bisect the string's half; this gives the fourth.
3) Trisect the string; the 2/3 point gives the fifth.
4) Bisect the segment between the fourth and the fifth; this gives the tritone or augmented fourth.
5) Divide the segment between the fourth and the prime into five arithmetically equal parts for the semitones between those notes.
6) Divide the segment between the fifth and the octave into five equal parts for the semitones between those notes.

Thus, if the full string length is 120, the divisions will fall at:
(Step 1) 60;
(Step 2) 90;
(Step 3) 80;
(Step 4) 85;
(Step 5) 96, 102, 108, and 114; (Step 6) 76, 72, 68, and 64.

The diatonic notes of the resulting scale are in just intonation, except that the major second (tonic to supertonic) has the ratio 9:10 rather than 8:9. The chromatic notes are indifferent approximations to equal temperament, and there are six different sizes of semitones, ranging from 89 to 112 cents.

A comparison with the pitches in the 17:18 tuning shows that the latter is closer to equal temperament, except in the intervals of the fourth, the tritone (or diminished fifth), the fifth, and the octave. This is seen in the following table, where for convenience the open string is assumed to produce C.

Note	String Length	Ratio to C	Cents	Cents for: 17:18	equal
C	120	1:1	0	0	0
±	114	19:20	88.80	98.95	100
D	108	9:10	182.40	197.91	200
±	102	17:20	281:36	296.86	300
E	96	4:5	386.31	395.82	400
F	90	3:4	498.04	494.77	500
±	85	17:24	597.00	593.73	600
G	80	2:3	701.96	692.68	700
±	76	19:30	790.76	791.64	800
A	72	3:5	884.36	890.59	900
±	68	17:30	983.31	989.55	1000
B	64	8:15	1088.27	1088.50	1100
C	60	1:2	1200	1187.46	1200

One should remember that the stretching of a lute string caused by its (left-hand) fingering will, for all practical purposes, obviate these deviations from equal temperament in both methods, as will the loss of tension resulting from plucking the string while playing.

Appendix II. Constructible Approximations to the Equal-tempered Semitone

The twelfth root of 2 equals, to 12 decimal places, 1.053946309436, and is by definition 100 cents, or an equal-tempered semitone. The following table lists a number of rational and quadratic constructible approximations to this value, along with their first proposers where known (in some

cases the approximation listed has been calculated from some interval other than the semitone). They are listed in order of decreasing error. The best ones are accurate enough to have pleased Mendelssohn's ear but of course not his eye.

|	Approximation	Decimal Value	Cents	Error (cents) in	
				semitone	octave
	18/17	1.058824	98.9546	− 1.05	− 12.5
	512/483	1.060041	100.9447	+ 0.94	+ 11.3
	53/50	1.060000	100.8771	+ 0.88	+ 10.5
	$16\sqrt{43/99}$	1.059788	100.5309	+ 0.53	+ 6.4
	$8/\sqrt{57}$	1.059626	100.2660	+ 0.27	+ 3.2
Mersenne 1636	125/118	1.059322	99.7695	− 0.23	− 2.8
Mersenne 1636	$\sqrt[8]{(3-\sqrt{2})}$	1.059328	99.7794	− 0.22	− 2.6
Hammond 1934	3692/3485*	1.059397	99.8927	− 0.11	− 1.3
Ellis 1885	89/84	1.059524	100.0992	+ 0.10	+ 1.2
Cahill 193?	$\sqrt{55/7}$	1.059457	99.9899	− 0.010	− 0.12
	196/185**	1.059459	99.9941	− 0.006	− 0.071

To these may be added Lambert's astonishingly accurate approximation (advocated by Kirnberger in Section 18 of his Kunst des reinen Satzes; Kirnberger claimed that one could tune a clavier in a few minutes by Lambert's method, without the aid of a monochord), produced by ascending seven pure fifths (of 3:2) and a pure major third (of 5:4), then reducing by four octaves; the resulting ratio, 10935:8192, is almost exactly an equal-tempered fourth above the starting note. (If eight rather than seven fifths are added to the major third before octave reduction, an approximate octave is obtained, which differs from the true octave by the schisma mentioned in n. 32.) The semitone between this value and $\sqrt{2}$ (the equal-tempered tritone, easily constructed) has the ratio

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* Laurens Hammond needed rational approximations for all the notes of the chromatic octave for his Hammond Electric Organ, which depended on integral gear ratios for producing the tones. A full list and discussion of his ratios can be found in Barbour (op. cit.), pp. 74-76. The semitone in the table above was calculated from his C and C$^{\#}$/Db; other semitones of the scale vary slightly from the tabulated value.

** To my knowledge, the 196/185 approximation has not been published previously.

1.0594629377, or 99.999744 cents, for an error of only 0.003 cents in the octave. It is curious and somewhat ironic that Kirnberger, who argued against equal temperament, should have publicized the most accurate approximation of all; Marpurg called this "Kirnbergerius contra Kirnbergerium".

Appendix III Mendelssohn's Figures
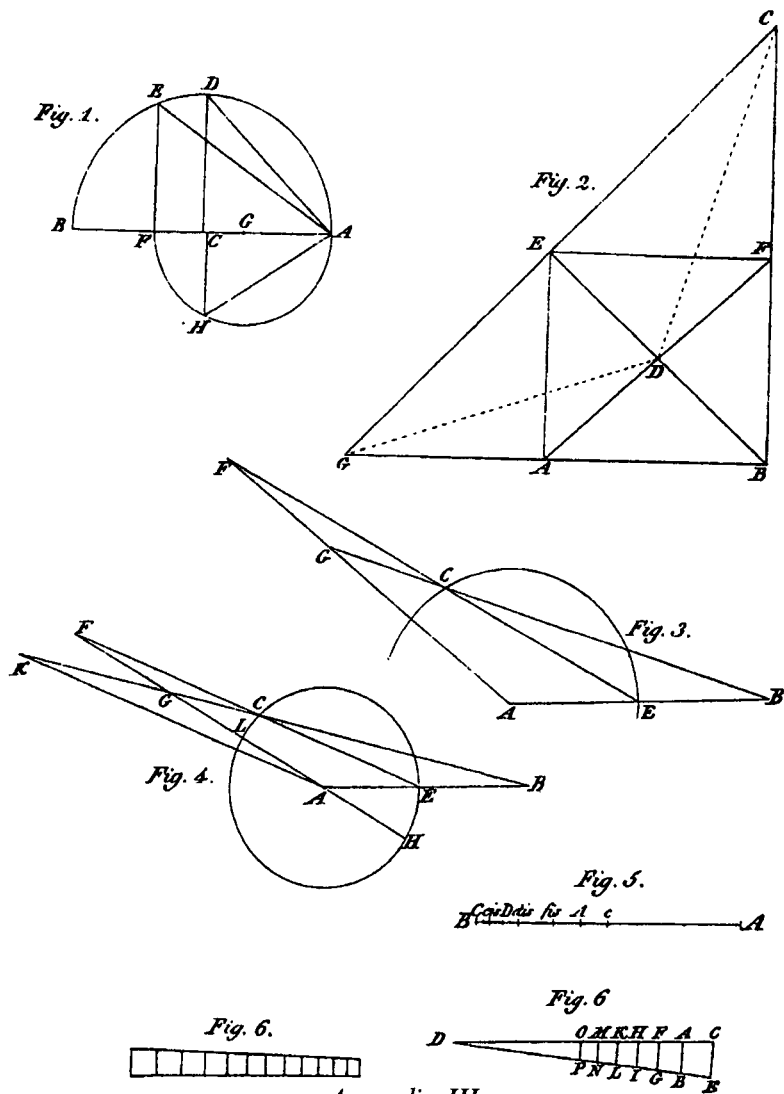
Appendix IIIa.
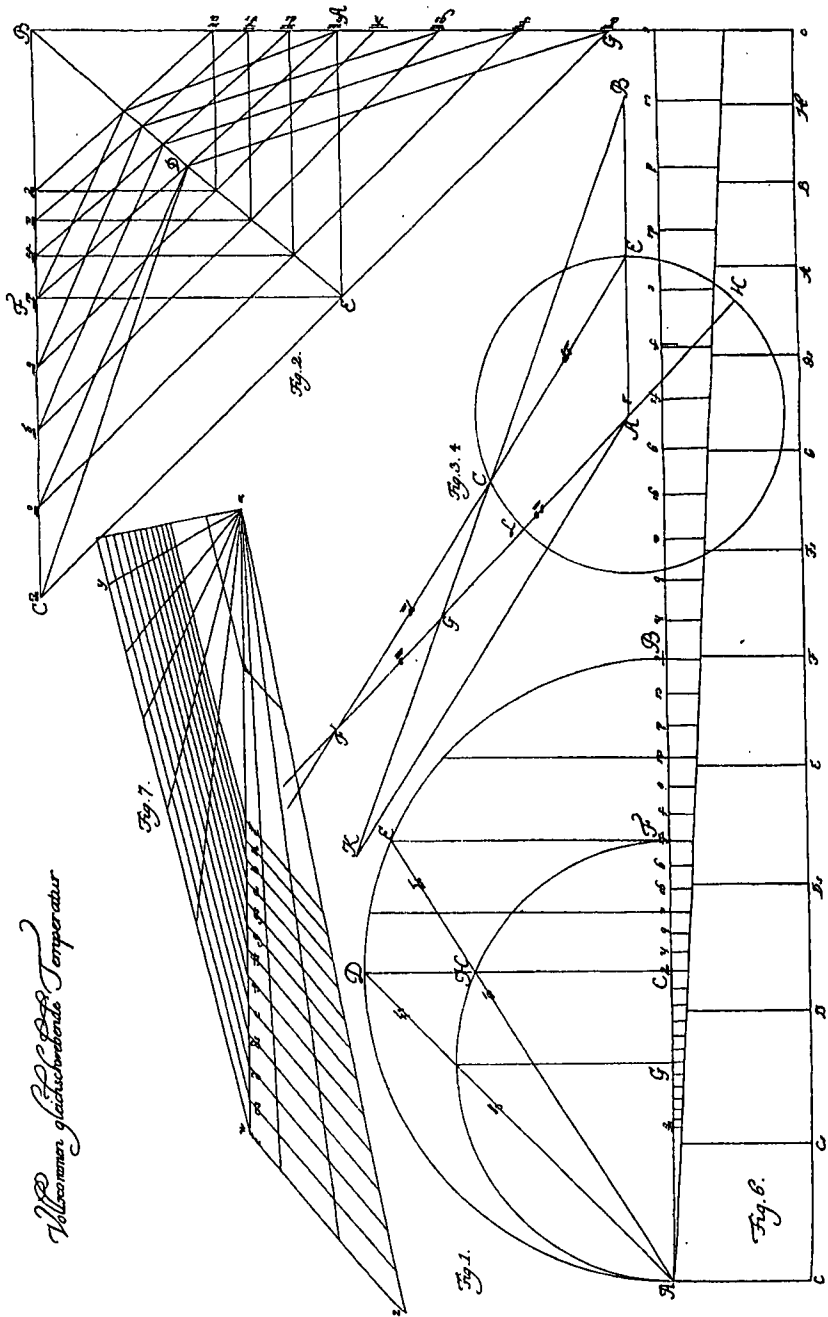
Appendix IIIb.

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